

Variants of the Busemann-Petty problem and of the Shephard problem

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Abstract

We provide an affirmative answer to a variant of the Busemann-Petty problem, proposed by V. Milman: Let K be a convex body in \mathbb{R}^n and let D be a compact subset of \mathbb{R}^n such that, for some $1 \leq k \leq n-1$,

$$|P_F(K)| \leq |D \cap F|$$

for all $F \in G_{n,k}$, where $P_F(K)$ is the orthogonal projection of K onto F and $D \cap F$ is the intersection of D with F . Then,

$$|K| \leq |D|.$$

We also provide estimates for the lower dimensional Busemann-Petty and Shephard problems, and we prove separation in the original Busemann-Petty problem.

1 Introduction

The Busemann-Petty problem was posed in [8], first in a list of ten problems concerning central sections of symmetric convex bodies in \mathbb{R}^n and coming from questions in Minkowski geometry. It was originally formulated as follows:

Assume that K and D are origin-symmetric convex bodies in \mathbb{R}^n and satisfy

$$(1.1) \quad |K \cap \xi^\perp| \leq |D \cap \xi^\perp|$$

for all $\xi \in S^{n-1}$. Does it follow that $|K| \leq |D|$?

Here ξ^\perp is the central hyperplane perpendicular to ξ . The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$ (for the history and the solution to this problem, see the monographs [12] and [18]). The isomorphic version of the Busemann-Petty problem asks if there exists an absolute constant $C_1 > 0$ such that whenever K and D satisfy (1.1) we have $|K| \leq C_1 |D|$. This question is equivalent to the slicing problem and to the isotropic constant conjecture asking if

$$(1.2) \quad L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\}$$

is a bounded sequence. More precisely, it is known that if K and D are two centered convex bodies in \mathbb{R}^n such that (1.1) holds true for all $\xi \in S^{n-1}$, then

$$(1.3) \quad |K|^{\frac{n-1}{n}} \leq c_1 L_n |D|^{\frac{n-1}{n}},$$

where $c_1 > 0$ is an absolute constant. Regarding L_n , Bourgain proved in [5] that $L_n \leq c\sqrt[4]{n} \log n$, and Klartag [16] improved this bound to $L_n \leq c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [17]. For more information on isotropic convex bodies and log-concave measures see [7].

Shephard's problem (see [30]) is dual to the Busemann-Petty problem.

Let K and D be two centrally symmetric convex bodies in \mathbb{R}^n . Suppose that

$$(1.4) \quad |P_{\xi^\perp}(K)| \leq |P_{\xi^\perp}(D)|$$

for every $\xi \in S^{n-1}$, where $P_{\xi^\perp}(A)$ is the orthogonal projection of $A \subset \mathbb{R}^n$ onto ξ^\perp . Does it follow that $|K| \leq |D|$?

The answer is affirmative if $n = 2$, but shortly after it was posed, Shephard's question was answered in the negative for all $n \geq 3$. This was done independently by Petty in [26] who gave an explicit counterexample in \mathbb{R}^3 , and by Schneider in [28] for all $n \geq 3$. After these counterexamples, one might try to relax the question, asking for the smallest constant C_n (or the order of growth of this constant C_n as $n \rightarrow \infty$) for which: if K, D are centrally symmetric convex bodies in \mathbb{R}^n and $|P_{\xi^\perp}(K)| \leq |P_{\xi^\perp}(D)|$ for all $\xi \in S^{n-1}$ then $|K| \leq C_n |D|$.

Such a constant C_n does exist, and a simple argument, based on John's theorem, shows that $C_n \leq c\sqrt{n}$, where $c > 0$ is an absolute constant. On the other hand, K. Ball has proved in [3] that this simple estimate is optimal: one has $C_n \simeq \sqrt{n}$.

In the first part of this note we discuss a variant of the two problems, proposed by V. Milman at the Oberwolfach meeting on Convex Geometry and its Applications (December 2015):

Question 1.1 (V. Milman). *Assume that K and D are origin-symmetric convex bodies in \mathbb{R}^n and satisfy*

$$(1.5) \quad |P_{\xi^\perp}(K)| \leq |D \cap \xi^\perp|$$

for all $\xi \in S^{n-1}$. Does it follow that $|K| \leq |D|$?

In Section 2 we show that the answer to this question is affirmative. In fact, the lower dimensional analogue of the problem has an affirmative answer. Moreover, one can drop the symmetry assumptions and even the assumption of convexity for D .

Theorem 1.2. *Let K be a convex body in \mathbb{R}^n and let D be a compact subset of \mathbb{R}^n such that, for some $1 \leq k \leq n-1$,*

$$(1.6) \quad |P_F(K)| \leq |D \cap F|$$

for all $F \in G_{n,n-k}$. Then,

$$(1.7) \quad |K| \leq |D|.$$

We also prove stability and separation in Theorem 1.2. In the hyperplane case, and assuming that K and D are centered convex bodies, i.e. their center of mass is at the origin, we can provide a more precise answer in terms of the isotropic constant L_D of D .

Theorem 1.3. *Let K and D be two centered convex bodies in \mathbb{R}^n such that*

$$(1.8) \quad |P_{\xi^\perp}(K)| \leq |D \cap \xi^\perp|$$

for all $\xi \in S^{n-1}$. Then,

$$(1.9) \quad |K| \leq \frac{c}{L_D} |D|,$$

where $c > 0$ is an absolute constant.

This means that if the hyperplane conjecture is not true then one can even have “pathologically good” (with respect to Question 1.1) pairs of convex bodies. The proof of Theorem 1.3 carries over to higher codimensions but the dependence on L_D becomes more complicated and we prefer not to include the full statement of this version.

In Section 3 we collect some estimates on the lower dimensional Busemann-Petty problem. Let $1 \leq k \leq n-1$ and let $\beta_{n,k}$ be the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in \mathbb{R}^n that satisfy

$$(1.10) \quad |K \cap F| \leq |D \cap F|$$

for all $F \in G_{n,n-k}$, one has

$$(1.11) \quad |K|^{\frac{n-k}{n}} \leq \beta^k |D|^{\frac{n-k}{n}}.$$

The following question is open:

Question 1.4. *Is it true that there exists an absolute constant $C_2 > 0$ such that $\beta_{n,k} \leq C_2$ for all n and k ?*

Bourgain and Zhang [6] showed that $\beta_{n,k} > 1$ if $n-k > 3$. It is not known whether $\beta_{n,k}$ has to be greater than 1 when $n \geq 5$ and $n-k = 2$ or $n-k = 3$. It was proved in [21] and by a different method in [10] that $\beta_{n,k} \leq C \sqrt{n/k} (\log(en/k))^{3/2}$, where C is an absolute constant. In this note, we observe that the answer to Question 1.4 is affirmative if the convex body K has bounded isotropic constant, as follows.

Theorem 1.5. *Let $1 \leq k \leq n-1$ and let K be a centered convex body in \mathbb{R}^n and D a compact subset of \mathbb{R}^n such that*

$$(1.12) \quad |K \cap F| \leq |D \cap F|$$

for all $F \in G_{n,n-k}$. Then,

$$(1.13) \quad |K|^{\frac{n-k}{n}} \leq (c_0 L_K)^k |D|^{\frac{n-k}{n}}.$$

where $c_0 > 0$ is an absolute constant.

Theorem 1.5 is a refinement of the estimate $\beta_{n,k} \leq cL_n$, which was shown in [10]. The proof is based on estimates from [11] and on Grinberg's inequality (see (2.4) in Section 2).

We also discuss the lower dimensional Shephard problem. Let $1 \leq k \leq n-1$ and let $S_{n,k}$ be the smallest constant $S > 0$ with the following property: For every pair of convex bodies K and D in \mathbb{R}^n that satisfy

$$(1.14) \quad |P_F(K)| \leq |P_F(D)|$$

for all $F \in G_{n,n-k}$, one has

$$(1.15) \quad |K|^{\frac{1}{n}} \leq S |D|^{\frac{1}{n}}.$$

Question 1.6. *Is it true that there exists an absolute constant $C_3 > 0$ such that $S_{n,k} \leq C_3$ for all n and k ?*

Goodey and Zhang [14] proved that $S_{n,k} > 1$ if $n-k > 1$. In Section 4 we prove the following result.

Theorem 1.7. *Let K and D be two convex bodies in \mathbb{R}^n such that*

$$(1.16) \quad |P_F(K)| \leq |P_F(D)|$$

for every $F \in G_{n,n-k}$. Then,

$$(1.17) \quad |K|^{\frac{1}{n}} \leq c_1 \sqrt{\frac{n}{n-k}} \log \left(\frac{en}{n-k} \right) |D|^{\frac{1}{n}},$$

where $c_1 > 0$ is an absolute constant. It follows that $S_{n,k}$ is bounded by an absolute constant if $\frac{k}{n-k}$ is bounded.

We also prove a general estimate, which is logarithmic in n and valid for all k . The proof is based on estimates from [25].

Theorem 1.8. *Let K and D be two convex bodies in \mathbb{R}^n such that*

$$(1.18) \quad |P_F(K)| \leq |P_F(D)|$$

for every $F \in G_{n,n-k}$. Then,

$$(1.19) \quad |K|^{\frac{1}{n}} \leq \frac{c_1 \min w(\tilde{D})}{\sqrt{n}} |D|^{\frac{1}{n}} \leq c_2 (\log n) |D|^{\frac{1}{n}},$$

where $c_1, c_2 > 0$ are absolute constants, $w(A)$ is the mean width of a centered convex body A , and the minimum is over all linear images \tilde{D} of D that have volume 1.

Lutwak [24] proved that the answer to the Busemann-Petty problem is affirmative if the body K with smaller sections belongs to a special class of intersection bodies; see definition below. In Section 5 we prove separation in the Busemann-Petty problem, which can be considered as a refinement of Lutwak's result.

Theorem 1.9. *Suppose that $\varepsilon > 0$, K and D are origin-symmetric star bodies in \mathbb{R}^n , K is an intersection body. If*

$$(1.20) \quad |K \cap \xi^\perp| \leq |D \cap \xi^\perp| - \varepsilon,$$

for every $\xi \in S^{n-1}$, then

$$|K|^{\frac{n-1}{n}} \leq |D|^{\frac{n-1}{n}} - c\varepsilon \frac{1}{\sqrt{n}M(\overline{K})},$$

where $c > 0$ is an absolute constant and $\overline{K} = |K|^{-\frac{1}{n}}K$.

Note that if \overline{K} is convex isotropic then

$$(1.21) \quad \frac{1}{M(\overline{K})} \geq c_1 \frac{n^{1/10}L_K}{\log^{2/5}(e+n)} \geq c_2 \frac{n^{1/10}}{\log^{2/5}(e+n)}$$

and if \overline{K} is convex and is in the minimal mean width position then we have

$$(1.22) \quad \frac{1}{M(\overline{K})} \geq c_3 \frac{\sqrt{n}}{\log(e+n)},$$

so the constant in Theorem 1.9 does not depend on the bodies. This is an improvement of a previously known result from [19]. Also note that stability in the Busemann-Petty problem is easier and was proved in [19], as follows. If K is an intersection body in \mathbb{R}^n , D is an origin-symmetric star body in \mathbb{R}^n and $\varepsilon > 0$ so that

$$(1.23) \quad |K \cap \xi^\perp| \leq |D \cap \xi^\perp| + \varepsilon$$

for every $\xi \in S^{n-1}$, then

$$(1.24) \quad |K|^{\frac{n-1}{n}} \leq |D|^{\frac{n-1}{n}} + c_n \varepsilon,$$

where $c_n = |B_2^{n-1}|/|B_2^n|^{\frac{n-1}{n}} < 1$. The constant is optimal. For more results on stability and separation in volume comparison problems and for applications of such results, see [20].

2 Milman's variant of the two problems

We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote the corresponding Euclidean norm by $\|\cdot\|_2$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . We also denote the Haar measure on $O(n)$ by ν . The Grassmann manifold $G_{n,m}$ of m -dimensional subspaces of \mathbb{R}^n is equipped with the Haar probability measure $\nu_{n,m}$. Let $1 \leq m \leq n-1$ and $F \in G_{n,m}$. We will denote the orthogonal projection from \mathbb{R}^n onto F by P_F . We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

A convex body in \mathbb{R}^n is a compact convex subset K of \mathbb{R}^n with nonempty interior. We say that K is origin-symmetric if $K = -K$. We say that K is centered if the center of mass of K is at the origin, i.e. $\int_K \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$. We denote by \mathcal{K}_n the class of centered convex bodies in \mathbb{R}^n . The support function of K is defined by $h_K(y) := \max\{\langle x, y \rangle : x \in K\}$, and the mean width of K is the average

$$(2.1) \quad w(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta)$$

of h_K on S^{n-1} . For basic facts from the Brunn-Minkowski theory and the asymptotic theory of convex bodies we refer to the books [29] and [1] respectively.

The proof of Theorem 1.2 is based on two classical results:

1. *Aleksandrov's inequalities.* If K is a convex body in \mathbb{R}^n then the sequence

$$(2.2) \quad Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F) \right)^{1/k}$$

is decreasing in k . This is a consequence of the Aleksandrov-Fenchel inequality (see [9] and [29]). In particular, for every $1 \leq k \leq n-1$ we have

$$(2.3) \quad \left(\frac{|K|}{\omega_n} \right)^{\frac{1}{n}} \leq \left(\frac{1}{\omega_k} \int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F) \right)^{\frac{1}{k}} \leq w(K),$$

where $w(K)$ is the mean width of K .

2. *Grinberg's inequality.* If D is a compact set in \mathbb{R}^n then, for any $1 \leq k \leq n-1$,

$$(2.4) \quad \tilde{R}_k(D) := \frac{1}{|D|^{n-k}} \int_{G_{n,n-k}} |D \cap F|^n d\nu_{n,n-k}(F) \leq \frac{1}{|B_2^n|^{n-k}} \int_{G_{n,n-k}} |B_2^n \cap F|^n d\nu_{n,n-k}(F),$$

where B_2^m is the Euclidean ball in \mathbb{R}^m and $\omega_m = |B_2^m|$. This fact was proved by Grinberg in [15]. It is useful to note that

$$(2.5) \quad \tilde{R}_k(B_2^n) := \frac{\omega_{n-k}^n}{\omega_n^{n-k}} \leq e^{\frac{kn}{2}}.$$

Moreover, Grinberg proved that the quantity $\tilde{R}_k(D)$ on the left hand side of (2.4) is invariant under $T \in GL(n)$: one has

$$(2.6) \quad \tilde{R}_k(T(D)) = \tilde{R}_k(D)$$

for every $T \in GL(n)$.

Proof of Theorem 1.2. Let K be a convex body in \mathbb{R}^n and D be a compact subset of \mathbb{R}^n . Assume that for some $1 \leq k \leq n-1$ we have

$$(2.7) \quad |P_F(K)| \leq |D \cap F|$$

for all $F \in G_{n,n-k}$. From (2.3) we get

$$(2.8) \quad \left(\frac{|K|}{\omega_n} \right)^{\frac{n-k}{n}} \leq \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |P_F(K)| d\nu_{n,n-k}(F).$$

Our assumption, Hölder's inequality and Grinberg's inequality give

$$(2.9) \quad \begin{aligned} \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |P_F(K)| d\nu_{n,n-k}(F) &\leq \frac{1}{\omega_{n-k}} \int_{G_{n,n-k}} |D \cap F| d\nu_{n,n-k}(F) \\ &\leq \frac{1}{\omega_{n-k}} \left(\int_{G_{n,n-k}} |D \cap F|^n d\nu_{n,n-k}(F) \right)^{\frac{1}{n}} \\ &\leq \frac{1}{\omega_{n-k}} \frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} |D|^{\frac{n-k}{n}} = \left(\frac{|D|}{\omega_n} \right)^{\frac{n-k}{n}}. \end{aligned}$$

Therefore, $|K| \leq |D|$. □

Remark 2.1. Slightly modifying the proof of Theorem 1.2 one can get stability and separation results, as follows. Let $\varepsilon > 0$, and let K and D be as in Theorem 1.2. Suppose that for every $F \in G_{n,n-k}$

$$|P_F(K)| \leq |D \cap F| \pm \varepsilon.$$

Then

$$|K|^{\frac{n-k}{n}} \leq |D|^{\frac{n-k}{n}} \pm \gamma_{n,k} \varepsilon,$$

where $\gamma_{n,k} = \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}} \in (e^{-k/2}, 1)$. The plus sign corresponds to stability, minus - to separation. Assuming that $\varepsilon = \max_F (|P_F(K)| - |D \cap F|)$ in the stability result, we get

$$|K|^{\frac{n-k}{n}} - |D|^{\frac{n-k}{n}} \leq \gamma_{n,k} \max_F (|P_F(K)| - |D \cap F|).$$

On the other hand, if $\varepsilon = \min_F (|D \cap F| - |P_F(K)|)$ in the separation result, then

$$|D|^{\frac{n-k}{n}} - |K|^{\frac{n-k}{n}} \geq \gamma_{n,k} \min_F (|D \cap F| - |P_F(K)|).$$

3 Estimates for the lower dimensional Busemann-Petty problem

In this section we provide some estimates for the lower dimensional Busemann-Petty problem. We need the next lemma, in which we collect known estimates about the quantities

$$(3.1) \quad G_{n,k}(A) := \left(\int_{G_{n,n-k}} |A \cap F|^n d\nu_{n,n-k}(F) \right)^{\frac{1}{kn}},$$

where A is a centered convex body in \mathbb{R}^n . The proofs of (3.2) and (3.3) can be found in [11], while (3.4) follows from (2.4) and (2.5).

Lemma 3.1. *Let A be a centered convex body in \mathbb{R}^n . Then,*

$$(3.2) \quad \frac{c_1}{L_A} |A|^{\frac{n-k}{kn}} \leq G_{n,k}(A) \leq \frac{c_2 L_k}{L_A} |A|^{\frac{n-k}{kn}} \leq \frac{c_3 \sqrt[4]{k}}{L_A} |A|^{\frac{n-k}{kn}}.$$

Moreover,

$$(3.3) \quad G_{n,k}(A) \leq c_4 \sqrt{n/k} (\log(en/k))^{\frac{3}{2}} |A|^{\frac{n-k}{kn}}.$$

Finally, for every compact subset D of \mathbb{R}^n we have

$$(3.4) \quad G_{n,k}(D) \leq \sqrt{e} |D|^{\frac{n-k}{kn}}.$$

Using Lemma 3.1 we show that the lower dimensional Busemann-Petty problem (Question 1.4) has an affirmative answer if the body K has bounded isotropic constant.

Proof of Theorem 1.5. Since $|K \cap F| \leq |D \cap F|$ for all $F \in G_{n,n-k}$, we know that

$$(3.5) \quad G_{n,k}(K) \leq G_{n,k}(D).$$

Using (3.2) and (3.4) we write

$$(3.6) \quad \frac{c_1}{L_K} |K|^{\frac{n-k}{kn}} \leq G_{n,k}(K) \leq G_{n,k}(D) \leq \sqrt{e} |D|^{\frac{n-k}{kn}},$$

and the result follows. \square .

Remark 3.2. Theorem 1.5 shows that if K belongs to the class

$$(3.7) \quad \mathcal{K}_n(\alpha) := \{K \in \mathcal{K}_n : L_K \leq \alpha\}$$

for some $\alpha > 0$, then for every compact set D in \mathbb{R}^n which satisfies $|K \cap F| \leq |D \cap F|$ for all $F \in G_{n,n-k}$ we have

$$(3.8) \quad |K|^{\frac{n-k}{n}} \leq (c_0 \alpha)^k |D|^{\frac{n-k}{n}}.$$

Classes of convex bodies with uniformly bounded isotropic constant include: unconditional convex bodies, convex bodies whose polar bodies contain large affine cubes, the unit balls of 2-convex spaces with a given constant α , bodies with small diameter (in particular, the class of zonoids) and the unit balls of the Schatten classes (see [7, Chapter 4]).

Example. K. Ball has proved in [2] that for every $1 \leq k \leq n-1$ and every $F \in G_{n,n-k}$ we have

$$(3.9) \quad |Q_n \cap F| \leq 2^{\frac{k}{2}},$$

where Q_n is the cube of volume 1 in \mathbb{R}^n . Consider the ball $B_{n,k} = r_{n,k} B_2^n$, where

$$(3.10) \quad \omega_{n-k} r_{n,k}^{n-k} = 2^{\frac{k}{2}}.$$

Then, for every $F \in G_{n,n-k}$ we have

$$(3.11) \quad |Q_n \cap F| \leq |B_{n,k} \cap F|.$$

Therefore,

$$(3.12) \quad 1 = |Q_n| \leq \beta_{n,k}^k |B_{n,k}|^{\frac{n-k}{n}} = \beta_{n,k}^k \omega_n^{\frac{n-k}{n}} r_{n,k}^{n-k} = 2^{\frac{k}{2}} \beta_{n,k}^k \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}}.$$

This proves that

$$(3.13) \quad \beta_{n,k} \geq \frac{1}{\sqrt{2}} \left(\frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} \right)^{\frac{1}{k}} \sim \frac{1}{\sqrt{2}} \left(\frac{n}{n-k} \right)^{\frac{n-k+1}{2k}}$$

as $n, k \rightarrow \infty$. Fix $d \geq 2$ and consider n and k that satisfy $n = (d+1)k$. Then, we have the following:

Proposition 3.3. *For every $d \geq 2$ there exists $k(d) \in \mathbb{N}$ such that*

$$(3.14) \quad \beta_{(d+1)k,k} \geq \frac{1}{\sqrt{2}} \left(1 + \frac{1}{d} \right)^{\frac{d}{2}} > 1$$

for all $k \geq k(d)$. □

A variant of the proof of Theorem 1.5 (based again on Lemma 3.1) establishes Theorem 1.3.

Proof of Theorem 1.3. Let K be a convex body in \mathbb{R}^n and D be a compact subset of \mathbb{R}^n such that $|P_{\xi^\perp}(K)| \leq |D \cap \xi^\perp|$ for every $\xi \in S^{n-1}$. From Lemma 3.1 we know that

$$(3.15) \quad G_{n,1}(D) \leq \frac{c_1}{L_D} |D|^{\frac{n-1}{n}},$$

where $c_1 > 0$ is an absolute constant. Then,

$$(3.16) \quad \begin{aligned} \left(\frac{|K|}{\omega_n} \right)^{\frac{n-1}{n}} &\leq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |P_{\xi^\perp}(K)| d\sigma(\xi) \leq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |D \cap \xi^\perp| d\sigma(\xi) \\ &= \frac{1}{\omega_{n-1}} \left(\int_{S^{n-1}} |D \cap \xi^\perp|^n d\sigma(\xi) \right)^{\frac{1}{n}} \\ &= \frac{1}{\omega_{n-1}} G_{n,1}(D) \leq \frac{c_1}{\omega_{n-1} L_D} |D|^{\frac{n-1}{n}}, \end{aligned}$$

which implies that

$$(3.17) \quad |K| \leq \frac{c_2 \omega_n}{(\omega_{n-1} L_D)^{\frac{n}{n-1}}} |D| \leq \frac{c_3}{L_D} |D|,$$

where $c_2, c_3 > 0$ are absolute constants. □

4 Estimates for the lower dimensional Shephard problem

In this section we discuss the lower dimensional Shephard problem. First, we recall some facts for the class of zonoids. A zonoid is a limit of Minkowski sums of line segments in the Hausdorff metric. Equivalently, a symmetric convex body Z is a zonoid if and only if its polar body is the unit ball of an n -dimensional subspace of an L_1 -space; i.e. if there exists a positive measure μ (the supporting measure of Z) on S^{n-1} such that

$$h_Z(x) = \|x\|_{Z^\circ} = \frac{1}{2} \int_{S^{n-1}} |\langle x, y \rangle| d\mu(y).$$

The class of origin-symmetric zonoids coincides with the class of projection bodies. Recall that the projection body ΠK of a convex body K is the symmetric convex body whose support function is defined by

$$h_{\Pi K}(\xi) = |P_{\xi^\perp}(K)|, \quad \xi \in S^{n-1}.$$

From Cauchy's formula

$$|P_{\xi^\perp}(K)| = \frac{1}{2} \int_{S^{n-1}} |\langle u, \xi \rangle| d\sigma_K(u),$$

where σ_K is the surface area measure of K , it follows that the projection body of K is a zonoid whose supporting measure is σ_K . Minkowski's existence theorem implies that, conversely, every zonoid is the projection body of some symmetric convex body in \mathbb{R}^n .

Zonoids play a central role in the study of the original Shephard problem: suppose that K is a convex body in \mathbb{R}^n and Z is a zonoid in \mathbb{R}^n , and that $|P_{\xi^\perp}(K)| \leq |P_{\xi^\perp}(Z)|$ for all $\xi \in S^{n-1}$. Then,

$$(4.1) \quad |K| \leq |Z|.$$

The proof involves writing $Z = \Pi D$ for some convex body D , using the identity $V_1(K, \Pi D) = V_1(D, \Pi K)$ (where $V_1(A, B)$ is the mixed volume $V(A, \dots, A, B)$), the hypothesis in the form $\Pi(K) \subseteq \Pi(Z)$, and the monotonicity of $V_1(D, \cdot)$, to write

$$|Z| = V_1(Z, Z) = V_1(Z, \Pi D) = V_1(D, \Pi Z) \geq V_1(D, \Pi(K)) = V_1(K, \Pi D) = V_1(K, Z) \geq |K|^{\frac{n-1}{n}} |Z|^{\frac{1}{n}},$$

where in the last step we also employ Minkowski's first inequality. This shows that $|Z| \geq |K|$.

Since any projection of a zonoid is a zonoid, using an inductive argument we can prove the following (for a detailed account on this topic, see [12, Chapter 4]).

Theorem 4.1. *Let K be a convex body and let Z be a zonoid in \mathbb{R}^n such that*

$$(4.2) \quad |P_F(K)| \leq |P_F(Z)|$$

for every $F \in G_{n,n-k}$. Then,

$$(4.3) \quad |K| \leq |Z|.$$

Using Theorem 4.1 and the fact that every ellipsoid is a zonoid, we can give a simple bound for the constants $S_{n,k}$.

Proposition 4.2. *For all n and $1 \leq k \leq n-1$ we have $S_{n,k} \leq c_0 \sqrt{n}$, where $c_0 > 0$ is an absolute constant.*

Proof. Let K and D be two convex bodies in \mathbb{R}^n such that $|P_F(K)| \leq |P_F(D)|$ for every $F \in G_{n,n-k}$. There exists an ellipsoid \mathcal{E} in \mathbb{R}^n such that $D \subseteq \mathcal{E}$ and $|\mathcal{E}|^{\frac{1}{n}} \leq c_0 \sqrt{n} |D|^{\frac{1}{n}}$, where $c_0 > 0$ is an absolute constant (for example, see [4] where a sharp estimate for c_0 is also given). Since $D \subseteq \mathcal{E}$, we have

$$(4.4) \quad |P_F(K)| \leq |P_F(D)| \leq |P_F(\mathcal{E})|$$

for all $F \in G_{n,n-k}$. Since \mathcal{E} is a zonoid, Theorem 4.1 implies that

$$(4.5) \quad |K|^{\frac{1}{n}} \leq |\mathcal{E}|^{\frac{1}{n}} \leq c_0 \sqrt{n} |D|^{\frac{1}{n}}.$$

This shows that $S_{n,k} \leq c_0 \sqrt{n}$. □

We can elaborate on this argument if we use Pisier's theorem from [27] on the existence of α -regular M -ellipsoids for symmetric convex bodies in \mathbb{R}^n (see [7, Theorem 1.13.3]):

Theorem 4.3 (Pisier). *For every $0 < \alpha < 2$ and every symmetric convex body A in \mathbb{R}^n , there exists an ellipsoid \mathcal{E}_α such that*

$$\max\{N(A, t\mathcal{E}_\alpha), N(\mathcal{E}_\alpha, tA)\} \leq \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t \geq 1$, where $c(\alpha)$ is a constant depending only on α and satisfies $c(\alpha) = O((2-\alpha)^{-\alpha/2})$ as $\alpha \rightarrow 2$.

Theorem 4.4. *Let $1 \leq m \leq n-1$ and let K and D be two convex bodies in \mathbb{R}^n such that*

$$(4.6) \quad |P_F(K)| \leq |P_F(D)|$$

for every $F \in G_{n,m}$. Then,

$$(4.7) \quad |K|^{\frac{1}{n}} \leq c_1 \sqrt{\frac{n}{m}} \log\left(\frac{en}{m}\right) |D|^{\frac{1}{n}},$$

where $c_1 > 0$ is an absolute constant.

Proof. Consider the difference body $D - D$ of D , and the ellipsoid \mathcal{E}_α from Theorem 4.3, where $\alpha \in (0, 2)$ will be chosen in the end, that corresponds to $A = D - D$. Note that

$$(4.8) \quad N(\mathcal{E}_\alpha, c(\alpha)^{1/\alpha}(D - D)) \leq e^n,$$

therefore

$$(4.9) \quad |\mathcal{E}_\alpha|^{\frac{1}{n}} \leq ec(\alpha)^{1/\alpha} |D - D|^{\frac{1}{n}}.$$

Since

$$(4.10) \quad N(P_F(D - D), P_F(t\mathcal{E}_\alpha)) \leq N(D - D, t\mathcal{E}_\alpha) \leq \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $F \in G_{n,m}$, we have

$$(4.11) \quad |P_F(D - D)| \leq \exp\left(\frac{c(\alpha)n}{t_{n,m,\alpha}^\alpha}\right) |P_F(t_{n,m,\alpha}\mathcal{E}_\alpha)| = |P_F(et_{n,m,\alpha}\mathcal{E}_\alpha)|$$

if we choose

$$(4.12) \quad t_{n,m,\alpha} = \left(\frac{c(\alpha)n}{m}\right)^{\frac{1}{\alpha}}.$$

Now, if we set $\mathcal{E} := et_{n,m,\alpha}\mathcal{E}_\alpha$, we have

$$(4.13) \quad |P_F(K)| \leq |P_F(D)| \leq |P_F(D - D)| \leq |P_F(\mathcal{E})|$$

for every $F \in G_{n,m}$, and since \mathcal{E} is a zonoid, Theorem 4.1 shows that $|K| \leq |\mathcal{E}|$. Using also (4.9) and the fact that $c(\alpha) = O((2 - \alpha)^{-\alpha/2})$, we get

$$(4.14) \quad |K|^{\frac{1}{n}} \leq et_{n,m,\alpha} |\mathcal{E}_\alpha|^{\frac{1}{n}} \leq e^2 t_{n,m,\alpha} c(\alpha)^{1/\alpha} |D - D|^{\frac{1}{n}} \leq \frac{c_1}{2 - \alpha} \left(\frac{n}{m}\right)^{\frac{1}{\alpha}} |D|^{\frac{1}{n}},$$

where $c_1 > 0$ is an absolute constant (we have also used the fact that $|D - D|^{\frac{1}{n}} \leq 4|D|^{\frac{1}{n}}$ by the Rogers-Shephard inequality). Choosing $\alpha = 2 - \frac{1}{\log(\frac{en}{m})}$ we get the result. \square

Remark 4.5. The lower dimensional Shephard problem is related to Lutwak's conjectures about the affine quermassintegrals: for every convex body K in \mathbb{R}^n and every $1 \leq m \leq n - 1$, the quantities

$$(4.15) \quad \Phi_{n-m}(K) = \frac{\omega_n}{\omega_m} \left(\int_{G_{n,m}} |P_F(K)|^{-n} d\nu_{n,m}(F) \right)^{-1/n},$$

were introduced by Lutwak in [22] (and Grinberg proved in [15] that these quantities are invariant under volume preserving affine transformations). Lutwak conjectured in [23] that the affine quermassintegrals satisfy the inequalities

$$(4.16) \quad \omega_n^j \Phi_i(K)^{n-j} \leq \omega_n^i \Phi_j(K)^{n-i}$$

for all $0 \leq i < j < n$, where we agree that $\Phi_0(K) = |K|$ and $\Phi_n(K) = \omega_n$. Most of the conjectures about the affine quermassintegrals remain open (see [12, Chapter 9] for more details and references). If true, they would imply the following (see also [11]): there exist absolute constants $c_1, c_2 > 0$ such that for every convex body K in \mathbb{R}^n and every $1 \leq m \leq n - 1$,

$$(4.17) \quad c_1 \sqrt{n/m} |K|^{\frac{1}{n}} \leq \left(\int_{G_{n,m}} |P_F(K)|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}} \leq c_2 \sqrt{n/m} |K|^{\frac{1}{n}}.$$

Assuming (4.17) we can give an affirmative answer to Question 1.6. Indeed, let K and D be two convex bodies in \mathbb{R}^n such that $|P_F(K)| \leq |P_F(D)|$ for every $F \in G_{n,n-k}$. We write

$$(4.18) \quad \begin{aligned} c_1 \sqrt{n/(n-k)} |K|^{\frac{1}{n}} &\leq \left(\int_{G_{n,n-k}} |P_F(K)|^{-n} d\nu_{n,n-k}(F) \right)^{-\frac{1}{(n-k)n}} \\ &\leq \left(\int_{G_{n,n-k}} |P_F(D)|^{-n} d\nu_{n,n-k}(F) \right)^{-\frac{1}{(n-k)n}} \leq c_2 \sqrt{n/(n-k)} |D|^{\frac{1}{n}}, \end{aligned}$$

and this shows that $|K|^{\frac{1}{n}} \leq (c_2/c_1) |D|^{\frac{1}{n}}$.

The left hand side of (4.17) was proved by Paouris and Pivovarov in [25]:

Theorem 4.6 (Paouris-Pivovarov). *Let A be a convex body in \mathbb{R}^n . Then,*

$$(4.19) \quad \left(\int_{G_{n,m}} |P_F(A)|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}} \geq c\sqrt{n/m} |A|^{\frac{1}{n}}.$$

Using this fact one can obtain the following.

Proposition 4.7. *Let $1 \leq m \leq n-1$ and let K and D be two convex bodies in \mathbb{R}^n such that*

$$(4.20) \quad |P_F(K)| \leq |P_F(D)|$$

for every $F \in G_{n,m}$. Then,

$$(4.21) \quad |K|^{\frac{1}{n}} \leq \frac{c \min w(\tilde{D})}{\sqrt{n}} |D|^{\frac{1}{n}},$$

where $c > 0$ is an absolute constant, $w(A)$ is the mean width of a centered convex body A , and the minimum is over all linear images \tilde{D} of D that have volume 1.

Proof. Our assumption implies that

$$(4.22) \quad \left(\int_{G_{n,m}} |P_F(K)|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}} \leq \left(\int_{G_{n,m}} |P_F(D)|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}}$$

By the linear invariance of $\Phi_{n-m}(D)$, for any $\tilde{D} = T(D)$ where $T \in GL(n)$ and $|\tilde{D}| = 1$, we have

$$(4.23) \quad \left(\int_{G_{n,m}} |P_F(D)|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}} = |D|^{\frac{1}{n}} \left(\int_{G_{n,m}} |P_F(\tilde{D})|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}}.$$

Now, using Hölder's inequality we write

$$(4.24) \quad \left(\int_{G_{n,m}} |P_F(\tilde{D})|^{-n} d\nu_{n,m}(F) \right)^{-\frac{1}{mn}} \leq \left(\int_{G_{n,m}} |P_F(\tilde{D})| d\nu_{n,m}(F) \right)^{\frac{1}{m}}$$

From Aleksandrov's inequalities we have

$$(4.25) \quad \left(\int_{G_{n,m}} |P_F(\tilde{D})| d\nu_{n,m}(F) \right)^{\frac{1}{m}} \leq \omega_m^{\frac{1}{m}} w(\tilde{D}) \leq c_2 \sqrt{n/m} \frac{w(\tilde{D})}{\sqrt{n}}.$$

Taking into account Theorem 4.6 we get

$$(4.26) \quad \begin{aligned} c\sqrt{n/m} |K|^{\frac{1}{n}} &\leq \left(\int_{G_{n,m}} |P_F(K)|^{-\frac{n}{m}} d\nu_{n,m}(F) \right)^{-\frac{1}{n}} \leq \left(\int_{G_{n,m}} |P_F(D)| d\nu_{n,m}(F) \right)^{\frac{1}{m}} \\ &= \left(\int_{G_{n,m}} |P_F(\tilde{D})| d\nu_{n,m}(F) \right)^{\frac{1}{m}} |D|^{\frac{1}{n}} \leq c_2 \sqrt{n/m} \frac{w(\tilde{D})}{\sqrt{n}} |D|^{\frac{1}{n}}, \end{aligned}$$

and the result follows. □

As a corollary we have:

Theorem 4.8. *Let $1 \leq m \leq n-1$ and let K and D be two convex bodies in \mathbb{R}^n such that*

$$(4.27) \quad |P_F(K)| \leq |P_F(D)|$$

for every $F \in G_{n,m}$. Then,

$$(4.28) \quad |K|^{\frac{1}{n}} \leq c(\log n) |D|^{\frac{1}{n}},$$

where $c > 0$ is an absolute constant.

Proof. If D is in the minimal mean width position, we have (see [1, Chapter 6])

$$(4.29) \quad w(\overline{D}) \leq c_1 \sqrt{n} \log n.$$

The result follows from Proposition 4.7. □

5 Separation in the Busemann-Petty problem

For the proof of Theorem 1.9 we need several definitions from convex geometry. A closed bounded set K in \mathbb{R}^n is called a star body if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K , and the Minkowski functional of K defined by

$$(5.1) \quad \|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on \mathbb{R}^n .

The radial function of a star body K is defined by

$$(5.2) \quad \rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n, x \neq 0.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x .

We use the polar formula for volume of a star body

$$(5.3) \quad |K| = \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta,$$

where $d\theta$ stands for the uniform measure on the sphere with density 1.

The class of intersection bodies was introduced by Lutwak in [24]. Let K, D be origin-symmetric star bodies in \mathbb{R}^n . We say that K is the intersection body of D and write $K = ID$ if the radius of K in every direction is equal to the $(n-1)$ -dimensional volume of the section of D by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$(5.4) \quad \begin{aligned} \rho_K(\xi) &= \|\xi\|_K^{-1} = |D \cap \xi^\perp| = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_D^{-n+1} d\theta \\ &= \frac{1}{n-1} R(\|\cdot\|_D^{-n+1})(\xi), \end{aligned}$$

where $R : C(S^{n-1}) \rightarrow C(S^{n-1})$ is the spherical Radon transform

$$(5.5) \quad Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) dx, \quad \text{for all } f \in C(S^{n-1}).$$

All bodies K that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies. A more general class of intersection bodies is defined as follows. If μ is a finite Borel measure on S^{n-1} , then the spherical Radon transform $R\mu$ of μ is defined as a functional on $C(S^{n-1})$ acting by

$$(5.6) \quad (R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \text{for all } f \in C(S^{n-1}).$$

A star body K in \mathbb{R}^n is called an *intersection body* if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ , as functionals on $C(S^{n-1})$, i.e.

$$(5.7) \quad \int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \text{for all } f \in C(S^{n-1}).$$

Intersection bodies played the key role in the solution of the Busemann-Petty problem.

Recall that $d\sigma(x) = dx/|S^{n-1}|$ is the normalized uniform measure on the sphere, and denote by

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma(x).$$

Proof of Theorem 1.9. By (5.4), the condition (1.20) can be written as

$$(5.8) \quad R(\|\cdot\|_K^{-n+1})(\xi) \leq R(\|\cdot\|_L^{-n+1})(\xi) + (n-1)\varepsilon, \quad \text{for all } \xi \in S^{n-1}.$$

Since K is an intersection body, there exists a finite Borel measure μ on S^{n-1} such that $\|\cdot\|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$. Together with (5.3), (5.8) and the definition of $R\mu$, the latter implies that

$$(5.9) \quad \begin{aligned} n|K| &= \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_K^{-n+1} dx = \int_{S^{n-1}} R(\|\cdot\|_K^{-n+1})(\xi) d\mu(\xi) \\ &\leq \int_{S^{n-1}} R(\|\cdot\|_L^{-n+1})(\xi) d\mu(\xi) - (n-1)\varepsilon \int_{S^{n-1}} d\mu(\xi) \\ (5.10) \quad &= \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx - (n-1)\varepsilon \int_{S^{n-1}} d\mu(x). \end{aligned}$$

We estimate the first term in (5.10) using Hölder's inequality:

$$(5.11) \quad \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx \leq \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{\frac{1}{n}} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{\frac{n-1}{n}} = n|K|^{\frac{1}{n}} |L|^{\frac{n-1}{n}}.$$

We now estimate the second term in (5.10) adding the Radon transform of the unit constant function under the integral ($R\mathbf{1}(x) = |S^{n-2}|$ for every $x \in S^{n-1}$), and using again the fact that $\|\cdot\|_K^{-1} = R\mu$:

$$(5.12) \quad \begin{aligned} (n-1)\varepsilon \int_{S^{n-1}} d\mu(x) &= \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} R\mathbf{1}(x) d\mu(x) = \frac{(n-1)\varepsilon}{|S^{n-2}|} \int_{S^{n-1}} \|x\|_K^{-1} dx \\ &\geq c_1 \varepsilon \frac{(n-1)|S^{n-1}|}{|S^{n-2}|} \frac{1}{M(\overline{K})} |K|^{\frac{1}{n}} \\ (5.13) \quad &\geq c_2 \varepsilon \sqrt{n} \frac{1}{M(\overline{K})} |K|^{\frac{1}{n}}, \end{aligned}$$

since

$$(5.14) \quad \int_{S^{n-1}} \|x\|_K^{-1} d\sigma(x) \geq \left(\int_{S^{n-1}} \|x\|_K d\sigma(x) \right)^{-1} = \frac{1}{M(\overline{K})} |K|^{\frac{1}{n}},$$

by Jensen's inequality, homogeneity, and

$$(5.15) \quad |S^{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \quad \text{and} \quad |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

Combining (5.13) with (5.10) and (5.11), we get

$$(5.16) \quad n|K| \leq n|K|^{\frac{1}{n}} |L|^{\frac{n-1}{n}} - c_2 \varepsilon \sqrt{n} \frac{1}{M(\overline{K})} |K|^{\frac{1}{n}}$$

and, after dividing by $n|K|^{1/n}$, the proof is complete. \square

Separation implies a volume difference inequality.

Corollary 5.1. *Let L be any origin-symmetric star body in \mathbb{R}^n , and let K be an origin-symmetric intersection body, which is a dilate of an isotropic body. Suppose that*

$$\min_{\xi \in S^{n-1}} (|L \cap \xi^\perp| - |K \cap \xi^\perp|) > 0.$$

Then

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} \geq c_2 \frac{1}{\sqrt{n} M(\overline{K})} \min_{\xi \in S^{n-1}} (|L \cap \xi^\perp| - |K \cap \xi^\perp|).$$

Remark 5.2. It was proved in [13] that there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any origin-symmetric isotropic convex body K in \mathbb{R}^n

$$(5.17) \quad \frac{1}{M(K)} \geq c_1 \frac{n^{1/10} L_K}{\log^{2/5}(e+n)} \geq c_2 \frac{n^{1/10}}{\log^{2/5}(e+n)}.$$

Also, if K is convex, has volume 1 and is in the minimal mean width position then we have

$$(5.18) \quad \frac{1}{M(K)} \geq c_3 \frac{\sqrt{n}}{\log(e+n)}.$$

Inserting these estimates into Theorem 1.9 and Corollary 5.1 we obtain estimates independent from the bodies.

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